# Optimisation

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September 22, 2020

[1] Slides based on the open access content of Martin J. Osborne

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# INTRODUCTION

## INTUITION

In economics, agents are assumed to be endowed with a **payoff function**, which is nothing else than an ordering of their preferences over the results of their actions.

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At the same time, agents are supposed to take **rational choices**, meaning that they maximised these payoff functions.

For example:

- Consumers are meant to maximise their utility over purchases
- Firms are supposed to maximise profits over investments
- Parties maximise votes over programmes
- and so on...

# INTRODUCTION

## DEFINITION

Let $f(\mathbf{x})$ be a function of many variables defined on a set $X$ and let $S$ be a subset of $X$. The point $\mathbf{x}^* \in S$ solves the problem

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$$f(\mathbf{x}) \leq f(\mathbf{x}^*) \quad \forall \mathbf{x} \in S$$

In this case we say that $\mathbf{x}^*$ is a **maximiser** of $f(\mathbf{x})$ subject to the constraint $\mathbf{x} \in S$, and that $f(\mathbf{x}^*)$ is the **maximum** (or maximum value) of $f(\mathbf{x})$ subject to the constraint $\mathbf{x} \in S$.

# INTRODUCTION

## LOCAL VS GLOBAL

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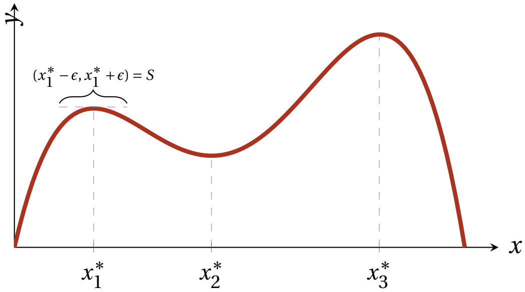


Local maximum around the interval $S$

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**REMARK:** This fact is useful since a function $f(\mathbf{x})$ can be transformed in such a way that the resulting function is easier to work with.

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$$v(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$$

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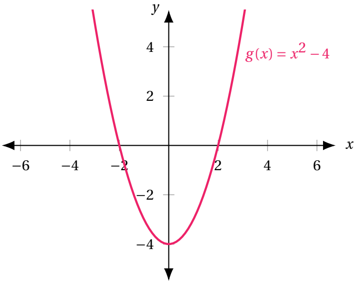


Minimisation problem

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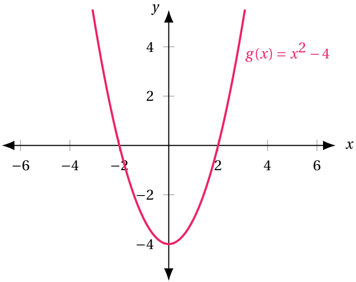


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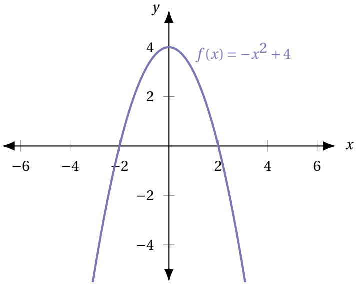


Maximisation problem

**EXTREME VALUE THEOREM:** let $f(\mathbf{x})$ be a **continuous** function defined on $X$ and let $S$ be a **compact** subset of $X$. Then the problems:

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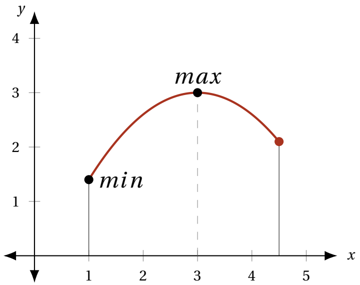

What if the conditions for an optimum are **relaxed**, i.e. are not met?:

**BOUNDEDNESS:** The set $S$ is bounded if there exists a number $k < \infty$ such that the distance of every point in S from the origin is at most $k$.

**Example:**

- Bounded set: $S = \{(x, y) \in \mathbb{R}^2 | 0 < x < 1, -10 \leq y < \pi/2\}$
- Unbounded set: $S = \{(x, y) \in \mathbb{R}^2 | 0 < x < \infty, -10 \leq y < \pi/2\}$

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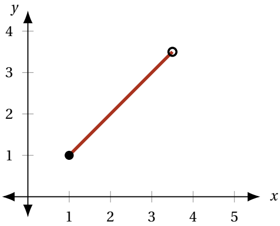


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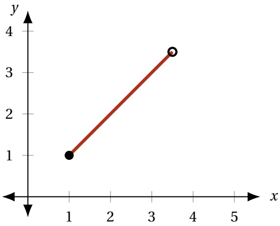


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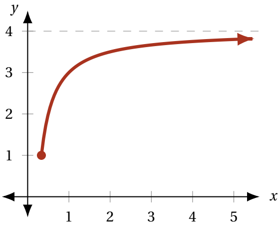


Unbounded

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- The set $S$ of n-vectors is open if every point in $S$ is an interior point of S.
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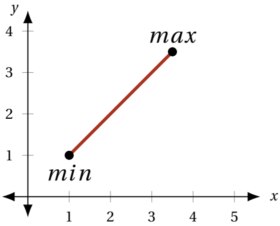


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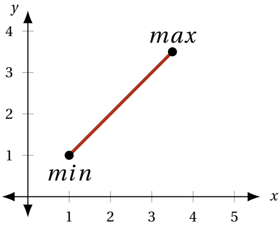
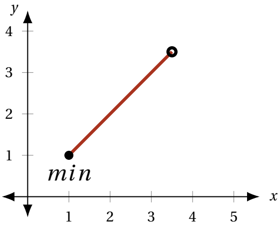


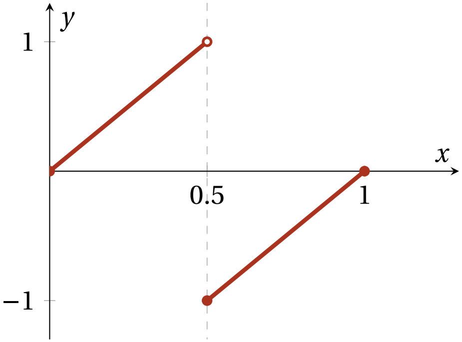

Closed




Open

**CONTINUITY:** a function is continuous if $\lim_{x \to a} f(x) = f(a)$

**Example:** relaxing continuity




Non-continuous function

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In this lecture we will see conditions to **maximise** in an interior point.

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**DEFINITION:** Let the function $f(\mathbf{x})$ be defined on a set $S$. A point $x \in S$ is a **stationary point** of $f(\mathbf{x})$ if $f(\mathbf{x})$ is differentiable and $f_i(\mathbf{x}) = 0$, for $i = 1, 2, \ldots$.

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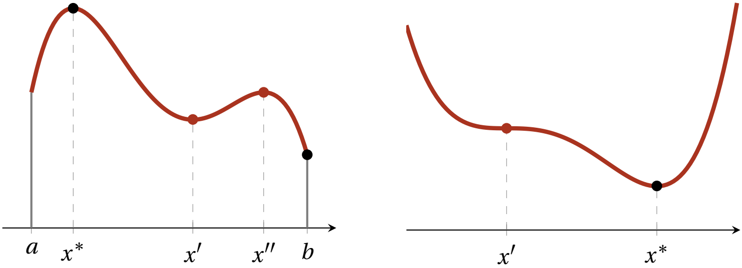


- In the left figure the points $x^*, x', x''$ are stationary points and extreme points. In the right figure $x'$ is a stationary point but not a extreme
- On the left picture $b$ is a extreme point but is not a stationary point

# INTERIOR OPTIMA

## INTRODUCTION

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Then why is it interesting if at all?

The only case in which a local maximiser is not a stationary point is when it is at the boundary of the set. That is, any **interior point** that is a maximiser must be a stationary point.

# INTERIOR OPTIMA

## FIRST ORDER CONDITIONS

**PROPOSITION:** Let $f(\mathbf{x})$ be defined on the set $S$. If $\mathbf{x}$ is a maximiser in the interior of $S$ and the partial derivatives exist w.r.t. the $i-th$ variable. Then:

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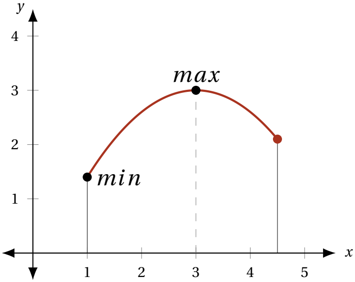
The first-derivative is involved, so we refer to the condition as a **first-order condition FOC**

# INTERIOR OPTIMA

## FIRST ORDER CONDITIONS

**PROOF:** Let the point $\mathbf{x}^*$ be a local maximiser, then it is clear that $f(x_1^* + h_1, \mathbf{x}_{-1}) \leq f(x_1^*, \mathbf{x}_{-1})$ for any $(x_1^* + h_1, \mathbf{x}_{-1}) \in S$, or in other words $f(x_1^* + h_1, \mathbf{x}_{-1}) - f(x_1^*, \mathbf{x}_{-1}) \leq 0$.

- Approaching the point from the right: $h > 0 \Rightarrow \lim_{h^+ \to 0} \frac{f(x_1^* + h_1, \mathbf{x}_{-1}) - f(x_1^*, \mathbf{x}_{-1})}{h} \leq 0$
- Approaching the point from the left: $h < 0 \Rightarrow \lim_{h^- \to 0} \frac{f(x_1^* + h_1, \mathbf{x}_{-1}) - f(x_1^*, \mathbf{x}_{-1})}{h} \geq 0$




Because the continuity of $f(x)$, there will be a point on $I$ such that $f'(x) = 0$

# INTERIOR OPTIMA

## FIRST ORDER CONDITIONS

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**IF:**

- $\mathbf{x}^*$ is a maximiser
- $x^*$ is in the interior of $S$
- $f_i$ exist $\forall i = 1, 2, \ldots$

**THEN:**

- $\mathbf{x}^*$ is a **Stationary Point**, i.e. $f_i'(\mathbf{x}^*) = 0 \; \forall i = 1, 2, \ldots$

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Let $f$ be a differentiable function of $n$ variables and let $S$ be a set of $n$-vectors. If the problem

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1. Use the **FOC** to find $\mathbf{x}^*$ and evaluate $f(\mathbf{x}^*)$
2. Along them find the values of the function at the boundary of $S$
3. The largest values of $f(\mathbf{x}^*)$ are the maximisers of $f$.

# INTERIOR OPTIMA

## FIRST ORDER CONDITIONS

**Example 1:** Consider the problem:

$$\begin{aligned} \max_{x,y} \quad & f(x,y) = -(x-1)^2 - (y+2)^2 \\ s.t. \quad & -\infty < x < \infty, \\ & -\infty < y < \infty \end{aligned}$$

The problem does not meet the conditions of the extreme value theorem —$x, y \in (-\infty, \infty)$ —so it is not possible to know beforehand if the problem will have a solution.

First order conditions:

$$\begin{aligned} f_x(x,y) &= -2(x-1) = 0 & \Rightarrow \quad & x^* = 1 \\ f_y(x,y) &= -2(y+2) = 0 & \Rightarrow \quad & y^* = -2 \end{aligned}$$

Then, the point $(1, -2)$ is stationary, we do not know yet if it is a maximiser.

# INTERIOR OPTIMA

## FIRST ORDER CONDITIONS

**Example 2:** Consider the problem:

$$\max_{x,y} \quad f(x,y) = (x-1)^2 + (y-1)^2$$
$$s.t. \quad 0 \le x \le 2,$$
$$-1 \le y \le 3$$

The problem does meet the conditions of the extreme value theorem - $x, y \in S$ - so it is possible to know beforehand that the problem will have maximum(a) and minimum(a).

First order conditions:

$$f_x(x,y) = 2(x-1) = 0 \quad \Rightarrow \quad x^* = 1$$
$$f_y(x,y) = 2(y-1) = 0 \quad \Rightarrow \quad y^* = 1$$

Then the point $(x^*, y^*) = (1,1)$ is stationary, where $f(x^*, y^*) = 0$

**Example 2:** Continuation:

Now consider the behaviour of the objective function on the boundary of the set $S$, which is a rectangle:

- Consider $x = 0$ and $-1 \leq y \leq 3$ then $f(0, y) = 1 + (y-1)^2$. By the FOC: $f_y(0, y^*) = 2(y-1) = 0 \Rightarrow y = 1$ which is in $int(S)$. Again we look at the boundary points in $\{(0, y) \in \mathbb{R}^2 | -1 \leq y \leq 3\}$, i.e. the points $(0, -1)$ and $(0, 3)$ are the candidates for optima where the value of the function is $f(0, -1) = f(0, 3) = 5$
- A similar analysis leads to points $(2, -1)$ and $(2, 3)$ being candidates for optima and where the function attains $f(2, -1) = f(2, 3) = 5$

Comparing the values of the function at the stationary points $(1, 1)$ and at the boundary points $(0, -1), (0, 3), (2, -1)$ and $(2, 3)$ we can conclude that the function has 4 solutions.

**Example 3:** Consider the problem:

$$\begin{cases} \max_{x,y} & f(x,y) = x^2 + y^2 + y - 1 \\ s.t. & x^2 + y^2 \leq 1 \end{cases} \text{ and } \begin{cases} \min_{x,y} & f(x,y) = x^2 + y^2 + y - 1 \\ s.t. & x^2 + y^2 \leq 1 \end{cases}$$

These problems meet the criteria of the extreme value theorem and hence they have solutions.

**FOC:**

$$\left.\begin{array}{r} f_x(x,y) = 2x = 0 \Rightarrow x^* = 0 \\ f_y(x,y) = 2y + 1 = 0 \Rightarrow y^* = -\frac{1}{2} \end{array}\right\} \Rightarrow (x^*, y^*) = \left(0, -\frac{1}{2}\right)$$

Then $\left(0, -\frac{1}{2}\right)$ is a stationary point where $f\left(0, -\frac{1}{2}\right) = -\frac{5}{4}$.

**Example 3:** Continuation

Turning to the boundary points we look at points that lay on the boundary, i.e. $x^2 + y^2 = 1$. Taking this equality into account the problem can be transform:

$$\text{from} \quad \begin{array}{cc} \max_{x,y} & f(x,y) = x^2 + y^2 + y - 1 \\ s.t. & x^2 + y^2 \leq 1 \end{array}$$

$$\text{into} \quad \begin{array}{cc} \max_{y} & f(y) = 1 + y - 1 = y \\ s.t. & 0 \leq y \leq 1 \end{array}$$

Clearly the minimum of this new problem is at $(1,0)$ and the maximum at $(0,1)$ where the functions attain 0 and 1 respectively.

Comparing the stationary and boundary points we see that the maximum is at $(0,1)$ and the minimum at $\left(0,-\frac{1}{2}\right)$

**MATHEMATICAL DETOUR:**

Let $f$ be a twice-differentiable function of $n$ variables. The **Hessian Matrix** of $f$ at $x$ is the matrix of second derivatives function, i.e.

$$\mathbf{H}(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_mx_1} & f_{x_mx_2} & \cdots & f_{x_mx_n} \end{pmatrix}$$

**NOTE:** because $f_{ij} = f_{ji}$ the matrix is symmetric.

# INTERIOR OPTIMA

## SECOND ORDER CONDITIONS

**PROPOSITION:** Let $f(\mathbf{x})$ be a twice-differentiable function with continuous partial derivatives and cross partial derivatives, defined on the set $S$. Suppose that $f_i(\mathbf{x}^*) = 0$, $\forall i$ for some $\mathbf{x}^*$ in the interior of $S$ (so that $\mathbf{x}^*$ is a stationary point of $f$). Let $\mathbf{H}$ be the Hessian of $f(\mathbf{x})$:

# INTERIOR OPTIMA

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- If $\mathbf{H}(\mathbf{x}^*)$ is negative definite then $\mathbf{x}^*$ is a local maximiser
- If $\mathbf{x}^*$ is a local maximiser then $\mathbf{H}(\mathbf{x}^*)$ is negative semi-definite
- If $\mathbf{H}(\mathbf{x}^*)$ is positive definite then $\mathbf{x}^*$ is a local minimiser
- If $\mathbf{x}^*$ is a local minimiser then $\mathbf{H}(\mathbf{x}^*)$ is positive semi-definite

# INTERIOR OPTIMA

## SECOND ORDER CONDITIONS

The previous slide implies that:

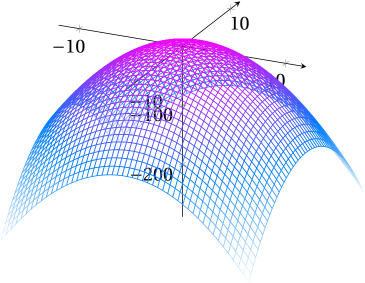
- If $\mathbf{H}(\mathbf{x}^*)$ is negative semi-definite Then $\mathbf{x}^*$ is either a maximiser or a saddle point
- If $\mathbf{H}(\mathbf{x}^*)$ is positive semi-definite Then $\mathbf{x}^*$ is either a minimiser or a saddle point

For this reason the determinant test should be summoned:

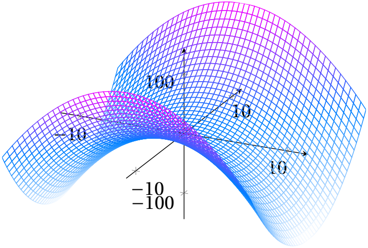
- If $|\mathbf{H}(\mathbf{x}^*)| < 0$ Then $\mathbf{x}^*$ is a saddle point
- If $|\mathbf{H}(\mathbf{x}^*)| > 0$ and $\mathbf{H}(\mathbf{x}^*)$ is n.s.d. Then $\mathbf{x}^*$ is a maximum point
- If $|\mathbf{H}(\mathbf{x}^*)| > 0$ and $\mathbf{H}(\mathbf{x}^*)$ is p.s.d. Then $\mathbf{x}^*$ is a minimum point
- If $|\mathbf{H}(\mathbf{x}^*)| = 0$ Then the test is inclusive. Solve by inspection

# INTERIOR OPTIMA

## SECOND ORDER CONDITIONS




$$f(x,y) = -x^2 - y^2$$




$$f(x,y) = -x^2 + y^2$$

**Example 1:** Consider the problem:

$$\max_{x,y} f(x,y) = x^3 + y^3 - 3xy$$

**FOC:**

$$\left.\begin{array}{l} f_x(x,y) = 3x^2 - 3y = 0 \Rightarrow x^2 = y \\ f_y(x,y) = 3y^2 - 3x = 0 \Rightarrow x = y^2 \end{array}\right\} \Rightarrow y = y^4 \text{ then } \left.\begin{array}{l} (x,y) = (0,0) \\ (x,y) = (1,1) \end{array}\right\}$$

**Example 1:**

Now the hessian of $f(x,y)$ at $(x,y)$ is:

$$\mathbf{H}(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

Turning to the hessian test:

1. $|\mathbf{H}(0,0)| = -9 < 0$ then is a saddle point
2. $|\mathbf{H}(1,1)| = 27 > 0$ also $f_{xx}(1,1) = 6$ and $f_{yy}(1,1) = 6$ and so the point is a local minimiser

**PROPOSITION:** Let $f(\mathbf{x})$ be a concave function defined on the convex set $S$, and let $x$ be in the interior of $S$.

Then

$\mathbf{x}^*$ is a **global** maximiser if and only if $\mathbf{x}^*$ is a stationary point of $f$, i.e. $f_i(\mathbf{x}) = 0 \; \forall \, i = 1, 2, \ldots, n$

# Table of Contents

# EQUALITY CONSTRAINTS

## INTUITION

**Example :** consider the problem

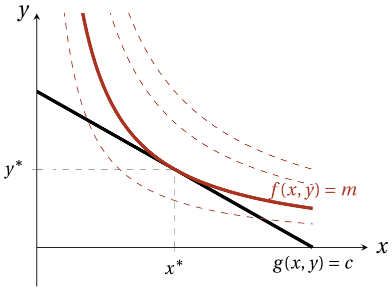
$$\max_{x,y} \quad f(x,y)$$
$$s.t. \quad g(x,y) = c$$

**Example :** consider the problem

$$\max_{x,y} \quad f(x,y)$$
$$s.t. \quad g(x,y) = c$$

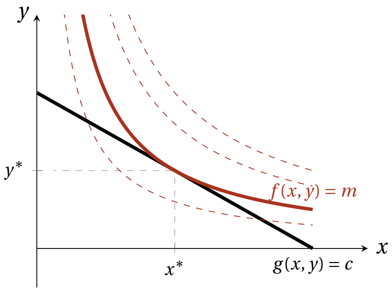
From the picture we see that the solution is where both lines are tangent, i.e.

$$-\frac{f_x(x^*,y^*)}{f_y(x^*,y^*)} = -\frac{g_x(x^*,y^*)}{g_y(x^*,y^*)}$$



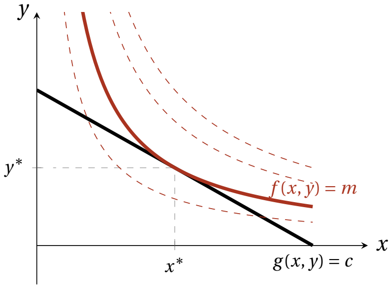
**Example continuation :** or rearranging

$$-\frac{f_x(x^*,y^*)}{g_x(x^*,y^*)} = -\frac{f_y(x^*,y^*)}{g_y(x^*,y^*)}$$



**Example continuation :** or rearranging

$$-\frac{f_x(x^*,y^*)}{g_x(x^*,y^*)} = -\frac{f_y(x^*,y^*)}{g_y(x^*,y^*)} = \lambda$$



# EQUALITY CONSTRAINTS

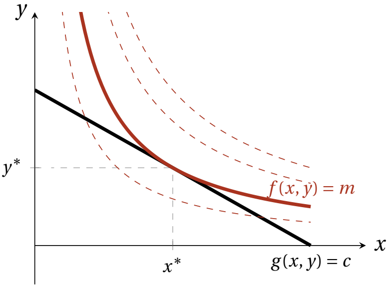
## INTUITION

**Example continuation :** or rearranging

$$-\frac{f_x(x^*, y^*)}{g_x(x^*, y^*)} = -\frac{f_y(x^*, y^*)}{g_y(x^*, y^*)} = \lambda$$

write this as two equations

$$f_x(x^*, y^*) - \lambda g_x(x^*, y^*) = 0$$
$$f_y(x^*, y^*) - \lambda g_y(x^*, y^*) = 0$$



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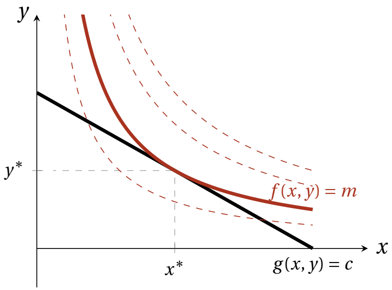



write this as two equations

$$f_x(x^*,y^*) - \lambda g_x(x^*,y^*) = 0$$
$$f_y(x^*,y^*) - \lambda g_y(x^*,y^*) = 0$$

We see that these conditions have to be satisfy at the solution, together with

$$c - g(x^*,y^*) = 0$$

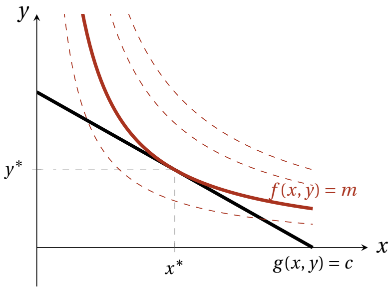
# EQUALITY CONSTRAINTS

## INTUITION

The first two equations can be viewed conveniently as the conditions for the derivatives of the Lagrangian

$$\mathscr{L}(x,y) = f(x,y) - \lambda\left\{g(x,y) - c\right\}$$

with respect to $x$ and $y$ to be zero.



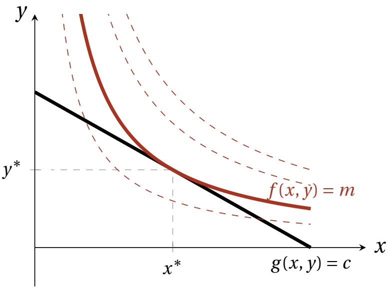
# EQUALITY CONSTRAINTS

## INTUITION

The first two equations can be viewed conveniently as the conditions for the derivatives of the Lagrangian

$$\mathscr{L}(x, y) = f(x, y) - \lambda \{g(x, y) - c\}$$

with respect to $x$ and $y$ to be zero. Known as the **FOC**



**PROPOSITION :** let $f(x,y)$ and $g(x,y)$ be continuously differentiable functions of two variables defined on the set $S$, let $c$ be a number, and assume $(x^*, y^*)$ is an interior point of $S$ that solves the problem:

$$\begin{array}{ll} \max_{x,y} & f(x,y) \\ s.t. & g(x,y) = c \end{array} \quad \text{or} \quad \begin{array}{ll} \min_{x,y} & f(x,y) \\ s.t. & g(x,y) = c \end{array}$$

Suppose also that either $g_x(x,y) \neq 0$ or $g_y(x,y) \neq 0$.

# EQUALITY CONSTRAINTS

## NECESSARY CONDITIONS

Then there is a unique number $\lambda$ such that $(x^*, y^*)$ is a stationary point of the **Lagrangian**:

$$\mathcal{L} = f(x,y) - \lambda\left(g(x,y) - c\right)$$

That is, $(x^*, y^*)$ satisfies the FOC:

$$\begin{aligned}
\mathcal{L}_x &= f_x(x,y) - \lambda g_x(x,y) = 0 \\
\mathcal{L}_y &= f_y(x,y) - \lambda g_y(x,y) = 0 \\
\mathcal{L}_\lambda &= g(x,y) - c = 0
\end{aligned}$$

# EQUALITY CONSTRAINTS

## NECESSARY CONDITIONS

**Example 1:** Consider the problem:

$$\max_{x,y} \quad xy$$
$$s.t. \quad x+y=6$$

Where the objective function $xy$ is defined on the set of all 2-vectors and the set $S$ is a line.

**Example 1:** Consider the problem:

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The Lagrangian is:

$$\mathscr{L}(x,y,\lambda) = xy - \lambda(x+y-6)$$

**Example 1:** Continuation:

FOC are:

$$\mathscr{L}_x(x, y, \lambda) = y - \lambda = 0$$
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$$\mathscr{L}_\lambda(x, y, \lambda) = x + y = 6$$

These equations have a unique solution $(x^*, y^*, \lambda^*) = (3, 3, 3)$. Also we have $g_x = 1 \neq 0$ and $g_y = 1 \neq 0$, $\forall (x, y)$, so if the problem has a solution it must be at $(3, 3)$

# EQUALITY CONSTRAINTS

## NECESSARY CONDITIONS

**Example 2:** Consider the problem:

$$\max_{x,y} \quad x^2 y$$
$$s.t. \quad 2x^2 + y^2 = 3$$

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# EQUALITY CONSTRAINTS

## NECESSARY CONDITIONS

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What does the **extreme value theorem** tell us about the solutions of this problem? The elipse is **compact**, hence theorem tell us that there is a solution.

The Lagrangian is:

$$\mathscr{L}(x, y, \lambda) = x^2 y - \lambda(2x^2 + y^2 - 3)$$

**Example 2:** Continuation:

FOC are:

# EQUALITY CONSTRAINTS

## NECESSARY CONDITIONS

**Example 2:** Continuation:
FOC are:

$$\mathscr{L}_x(x,y,\lambda) = 2x(y-2\lambda) = 0 \tag{1}$$

$$\mathscr{L}_y(x,y,\lambda) = x^2 - 2\lambda y = 0 \tag{2}$$

$$\mathscr{L}_\lambda(x,y,\lambda) = 2x^2 + y^2 - 3 = 0 \tag{3}$$

**Example 2:** Continuation:
FOC are:

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$$\mathscr{L}_y(x, y, \lambda) = x^2 - 2\lambda y = 0 \tag{2}$$

$$\mathscr{L}_\lambda(x, y, \lambda) = 2x^2 + y^2 - 3 = 0 \tag{3}$$

To find the solutions to the system of equations notice that to meet the first equation either $x = 0$ or $y = 2\lambda$.

**Example 2:** Continuation:

In turns:

- If $x = 0$, then (3) implies $y = \pm\sqrt{3}$ and (2) resulst in $\lambda = 0$.

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    - If $x = -y$, plugging this into (3) results in $3x^2 = 3 \Leftrightarrow x = \pm 1$ and as a result $y = \mp 1$

Then the possible **solutions** are:

$$(0, \sqrt{3}, 0) \text{ with } f(0, \sqrt{3}) = 0 \qquad (0, -\sqrt{3}, 0) \text{ with } f(0, -\sqrt{3}) = 0$$

$$\left(1, 1, \frac{1}{2}\right) \text{ with } f(1, 1) = 1 \qquad \left(-1, -1, -\frac{1}{2}\right) \text{ with } f(-1, -1) = -1$$

$$\left(1, -1, -\frac{1}{2}\right) \text{ with } f(1, -1) = -1 \qquad \left(-1, 1, \frac{1}{2}\right) \text{ with } f(-1, 1) = 1$$

**Example 2:** Continuation:

Now $g_x = 4x$ and $g_y = 2y$, the only value in which $g_x = g_y = 0$ is $(0,0)$. At this point the constraint is not satisfied, thus the only solutions are the ones that meet the FOC.

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Now $g_x = 4x$ and $g_y = 2y$, the only value in which $g_x = g_y = 0$ is $(0,0)$. At this point the constraint is not satisfied, thus the only solutions are the ones that meet the FOC.

Since it is a maximisation problem we can safely conclude that the only solution is $(x, y) = (1,1)$ and $(x, y) = (-1,1)$

# EQUALITY CONSTRAINTS

## LAGRANGE MULTIPLIERS

**INTUITION:** the value of the **Lagrange multiplier** at the solution of the problem is equal to the rate of change in the maximal value of the objective function as the constraint is relaxed.

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**Example:** Consider the problem

$$\max_{x} \quad x^2$$
$$s.t. \quad x = c$$

The solution of this problem is obvious: $x = c$. The maximised value of the function is thus $c^2$, so that the derivative of this maximised value with respect to $c$ is $2c$.

# INTERIOR OPTIMA

## LAGRANGE MULTIPLIERS

Let's check that the value of the Lagrange multiplier at the solution of the problem is equal to $2c$. The Lagrangian is:

$$\mathscr{L}(x) = x^2 - \lambda(x - c)$$

# INTERIOR OPTIMA

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$$\mathscr{L}(x) = x^2 - \lambda(x - c)$$

so the first-order condition is

$$2x - \lambda = 0$$

The constraint is $x = c$, so the pair $(x, \lambda)$ that satisfies the first-order condition and the constraint is $(c, 2c)$. Thus we see that indeed $\lambda$ is equal to the derivative of the maximised value of the function with respect to $c$.

# EQUALITY CONSTRAINTS

## SUFFICIENT CONDITIONS

**PROPOSITION:** Let $f(x, y)$ and $g(x, y)$ be twice differentiable functions of two variables defined on the set $S$ and let $c$ be a number. Suppose that $(x*, y*)$, an interior point of $S$, and the number $\lambda^*$ satisfy the first-order conditions:

$$\begin{aligned} f_x(x^*, y^*) - \lambda^* g_x(x^*, y^*) &= 0 \\ f_y(x^*, y^*) - \lambda^* g_y(x^*, y^*) &= 0 \\ g(x^*, y^*) &= c \end{aligned}$$

Then:

- If $\mathbf{D}\left(x^*, y^*, \lambda^*\right) > 0$ then $(x^*, y^*)$ is a local maximiser
- If $\mathbf{D}\left(x^*, y^*, \lambda^*\right) < 0$ then $(x^*, y^*)$ is a local minimiser

**DEFINITION:** the determinant $\mathbf{D}\left(x^*, y^*, \lambda^*\right)$ is called the **Bordered Hessian of the Lagrangian** and takes the following form:

$$\mathbf{D}\left(x^*, y^*, \lambda^*\right) = \begin{vmatrix} \mathscr{L}_{\lambda\lambda} & \mathscr{L}_{\lambda x} & \mathscr{L}_{\lambda y} \\ \mathscr{L}_{x\lambda} & \mathscr{L}_{xx} & \mathscr{L}_{xy} \\ \mathscr{L}_{y\lambda} & \mathscr{L}_{yx} & \mathscr{L}_{yy} \end{vmatrix} = \begin{vmatrix} 0 & g_x & g_y \\ g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{vmatrix}$$

With this in mind we can state the following result

**Example:** Continuation of the previous one in "NECESSARY CONDITIONS":

The possible solutions where worked out:

$$(0,\sqrt{3},0) \text{ with } f(0,\sqrt{3}) = 0 \qquad (0,-\sqrt{3},0) \text{ with } f(0,-\sqrt{3}) = 0$$

$$\left(1,1,\frac{1}{2}\right) \text{ with } f(1,1) = 1 \qquad \left(-1,-1,-\frac{1}{2}\right) \text{ with } f(-1,-1) = -1$$

$$\left(1,-1,-\frac{1}{2}\right) \text{ with } f(1,-1) = -1 \qquad \left(-1,1,\frac{1}{2}\right) \text{ with } f(-1,1) = 1$$

It seems obvious that the points $(1,1)$ and $(-1,1)$ where global maximisers and the points $(1,-1)$ and $(-1,-1)$ where global minimisers.

But what about $(0,\sqrt{3})$ and $(0,-\sqrt{3})$? They are neither optima, are they local optima?

# EQUALITY CONSTRAINTS

## SUFFICIENT CONDITIONS

**Example Continuation:** set the bordered hessian of the lagrangian and calculate the determinant.

**Example Continuation:** set the bordered hessian of the lagrangian and calculate the determinant.

The determinant of the bordered hessian of the Lagrangian is in general:

$$\mathbf{D}(x, y, \lambda) = \begin{vmatrix} 0 & 4x & 2y \\ 4x & 2y - 4\lambda & 2x \\ 2y & 2x & -2\lambda \end{vmatrix} = 8\left[2\lambda\left(2x^2 + y^2\right) + y\left(4x^2 - y^2\right)\right]$$

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And at the solutions:

- $\left|D(0, \sqrt{3}, 0)\right| = -8 \cdot 3^{\frac{3}{2}}$, and then $(0, \sqrt{3}, 0)$ is a local minimiser
- $\left|D(0, -\sqrt{3}, 0)\right| = 8 \cdot 3^{\frac{3}{2}}$, and then $(0, -\sqrt{3}, 0)$ is a local maximiser

# EQUALITY CONSTRAINTS

## n VARIABLES AND m CONSTRAINTS

The Lagrangian method can easily be generalised to a problem of the form:

$$\max_{\mathbf{x}} f(\mathbf{x})$$
$$s.t.\ g_j(\mathbf{x}) = c_j \text{ for } j = 1, \ldots, m$$

where $\mathbf{x} = (x_1, \ldots, x_n)$.

Ending with a problem of $n$ variables and $m$ constraints.

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where $\mathbf{x} = (x_1, \ldots, x_n)$.

Ending with a problem of $n$ variables and $m$ constraints.

The Lagrangian for this problem is:

$$\mathscr{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left(g_j(\mathbf{x}) - c_j\right)$$

That is, there is one Lagrange multiplier for each constraint.

# EQUALITY CONSTRAINTS

## n VARIABLES AND m CONSTRAINTS

**DEFINITION:** For $j = 1, ..., m$ let $g_j(\mathbf{x})$ be a differentiable function of $n$ variables. The **Jacobian Matrix** of $(g_1, ..., g_m)$ at the point $x$ is:

$$\begin{pmatrix} g_{1x_1}(\mathbf{x}) & ... & g_{1x_n}(\mathbf{x}) \\ ... & ... & ... \\ g_{mx_1}(\mathbf{x}) & ... & g_{mx_n}(\mathbf{x}) \end{pmatrix}$$

# EQUALITY CONSTRAINTS

## n VARIABLES AND m CONSTRAINTS

**PROPOSITION:** Let $f(\mathbf{x})$ and $g_j(\mathbf{x}) = c_j$ for $j = 1, \ldots, m$ be continuously differentiable functions of $n$ variables defined on the set $S$, with $m \leq n$, let $c_j$ for $j = 1, \ldots, m$ be numbers, and suppose that $\mathbf{x}^*$ is an interior point of $S$ that solves the problem:

$$\max_{\mathbf{x}} f(\mathbf{x})$$
$$s.t.\ g_j(\mathbf{x}) = c_j \text{ for } j = 1, \ldots, m$$

or the problem

$$\min_{\mathbf{x}} f(\mathbf{x})$$
$$s.t.\ g_j(\mathbf{x}) = c_j \text{ for } j = 1, \ldots, m$$

Suppose also that the rank of the Jacobian matrix of $(g_1, \ldots, g_m)$ at the point $x^*$ is $m$.

# EQUALITY CONSTRAINTS

## n VARIABLES AND m CONSTRAINTS

Then there exist unique numbers $\lambda_1, \ldots, \lambda_m$ such that $x^*$ is a stationary point of the Lagrangian function $L$ defined by:

$$\mathscr{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left( g_j(\mathbf{x}) - c_j \right)$$

That is, $\mathbf{x}^*$ satisfies the FOC:

$$\mathscr{L}_i(\mathbf{x}) = f_i(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j g_{j x_i}(\mathbf{x}) = 0 \text{ for } i = 1, \ldots, n$$

In addition, $g_j(\mathbf{x}^*) = c_j$ for $j = 1, \ldots, m$

# EQUALITY CONSTRAINTS

## n VARIABLES AND m CONSTRAINTS

**Example:** Consider the problem:

$$\begin{aligned} \min_{x,y,z} \quad & x^2 + y^2 + z^2 \\ s.t. \quad & x + 2y + z = 1 \\ & 2x - y - 3z = 4 \end{aligned}$$

**Example:** Consider the problem:

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The Lagrangian is:

$$\mathscr{L}(x, y, z) = x^2 + y^2 + z^2 - \lambda_1 \left(x + 2y + z - 1\right) - \lambda_2 \left(2x - y - 3z - 4\right)$$

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$$\mathscr{L}(x,y,z) = x^2 + y^2 + z^2 - \lambda_1 \left(x + 2y + z - 1\right) - \lambda_2 \left(2x - y - 3z - 4\right)$$

This function is convex for any values of $\lambda_1$ and $\lambda_2$, so that any interior stationary point is a solution of the problem. Further, the rank of the Jacobian matrix is 2 (a fact you can take as given), so any solution of the problem is a stationary point. Thus the set of solutions of the problem coincides with the set of stationary points.

**Example:** Continuation:

FOC are:

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$$2x - \lambda_1 - 2\lambda_2 = 0 \quad (1)$$

$$2y - 2\lambda_1 + \lambda_2 = 0 \quad (2)$$

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$$x + 2y + z = 1 \quad (4)$$

$$2x - y - 3z = 4 \quad (5)$$

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$$x + 2y + z = 1 \quad (4)$$

$$2x - y - 3z = 4 \quad (5)$$

Solving (1) and (2) for $\lambda_1$ and $\lambda_2$ gives:

$$\lambda_1 = \frac{2}{5}x + \frac{4}{5}y \quad (6)$$

$$\lambda_2 = \frac{4}{5}x + \frac{2}{5}y \quad (7)$$

**Example:** Continuation:

Now substitute (6) and (7) into (3) and solve the system of equations:

$$x = \frac{16}{15},\ y = \frac{1}{3},\ z = -\frac{11}{15},\ \lambda_1 = \frac{52}{75} \text{ and } \lambda_2 = \frac{54}{75}$$

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$$x = \frac{16}{15},\ y = \frac{1}{3},\ z = -\frac{11}{15},\ \lambda_1 = \frac{52}{75} \text{ and } \lambda_2 = \frac{54}{75}$$

Then we can conclude that $(x, y, z) = \left(\frac{15}{16}, \frac{1}{3}, -\frac{11}{15}\right)$ is the unique solution to the problem.

# EQUALITY CONSTRAINTS

## ENVELOPE THEOREM

**Example:** Consider the following function $f(x;\mathbf{r}) = x^{r_1} - r_2 x$ where $0 < r_1 < 1$. Which has a maximisation point at:

**Example:** Consider the following function $f(x;\mathbf{r}) = x^{r_1} - r_2 x$ where $0 < r_1 < 1$. Which has a maximisation point at:

Using the FOC:

$$x^* = \left(\frac{r_1}{r_2}\right)^{\frac{1}{1-r_1}}$$

The value of the function at that point is

$$f(x^*;\mathbf{r}) = \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{1-r_1}} - r_2\left(\frac{r_1}{r_2}\right)^{\frac{1}{1-r_1}}$$

It might be interesting to know the effect of $r_1$ in the change of the value function. Can you try it?

**PROPOSITION:** Let $f(\mathbf{x};\mathbf{r})$ be a function of $n$ variables, let $\mathbf{r}$ be a $h$-vector of parameters, and let the $n$-vector $\mathbf{x}^*$ be a maximiser of $f(\mathbf{x};\mathbf{r})$. Assume that the partial derivative $f'_{n+k}(\mathbf{x}^*,\mathbf{r})$ (i.e. the partial derivative of $f(\mathbf{x};\mathbf{r})$ with respect to $\mathbf{r}_k$) at $(x^*,r)$ exists. Define the **Value Function** $f^*(\mathbf{r})$ of $k$ variables by:

$$f^*(\mathbf{r}) = \max_x f(\mathbf{x};\mathbf{r}), \quad \forall r_k.$$

If the partial derivative $f^*_k(\mathbf{r})$ exists then

$$f^*_k(\mathbf{r}) = f_{n+k}(\mathbf{x}^*,\mathbf{r}).$$

For $k = \{1,...,h\}$

# EQUALITY CONSTRAINTS

## ENVELOPE THEOREM

**INTUITION:** we might be interested in seeing how the function at the solution $f(\mathbf{x}^*;\mathbf{r})$ changes as some parameters $\mathbf{r}$ change.

# EQUALITY CONSTRAINTS

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**RESULT:** At the optimum only direct effects of the parameters into the function need taking into account, the indirect effects can be neglected since:

$$\frac{\partial f(\mathbf{x}^*(\mathbf{r});\mathbf{r})}{\partial r_k} = \frac{\partial f(\mathbf{x}^*(\mathbf{r});\mathbf{r})}{\partial x_i^*(\mathbf{r})} \cdot \frac{\partial x_i^*(\mathbf{r})}{\partial r_k} + \frac{\partial f^*(\mathbf{r})}{\partial r_k}$$

**INTUITION:** we might be interested in seeing how the function at the solution $f(\mathbf{x}^*;\mathbf{r})$ changes as some parameters $\mathbf{r}$ change.

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But at the optimum $\frac{\partial f(\mathbf{x}^*;\mathbf{r})}{\partial x_i^*} = 0$

Hence the result

# EQUALITY CONSTRAINTS

## ENVELOPE THEOREM

**Example:** Consider the following function $f(x;\mathbf{r}) = x^{r_1} - r_2 x$ where $0 < r_1 < 1$. Which has a maximisation point at:

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It might be interesting to know the effect of $r_1$ in the change of the value function. Thus by the envelope theorem:

$$\frac{\partial f\left(x^*(\mathbf{r});\mathbf{r}\right)}{\partial r_1} = \left(x^*(\mathbf{r})\right)^{r_1} \ln x^*(r)$$

# EQUALITY CONSTRAINTS

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$$\frac{\partial f(x^*(\mathbf{r});\mathbf{r})}{\partial r_1} = \left(x^*(\mathbf{r})\right)^{r_1} \ln x^*(r)$$

or substituting $x^*(\mathbf{r})$

$$\frac{\partial f(x^*(\mathbf{r});\mathbf{r})}{\partial r_1} = \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{1-r_1}} \frac{1}{1-r_1} \ln\left(\frac{r_1}{r_2}\right)$$

# Table of Contents

# INEQUALITY CONSTRAINTS

## INTUITION

Many models in economics are formulated as problems with inequality constraints.

Consider the consumer set of choices, we do not need to oblige the consumer to spend all of his budget, some of it might be saved.

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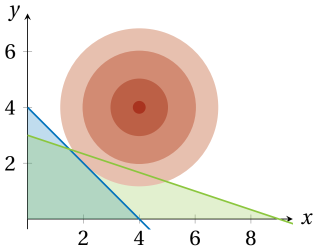
$$
\begin{aligned}
\max_{\mathbf{x}} \quad & u(\mathbf{x}) \\
s.t. \quad & \mathbf{p}\cdot\mathbf{x} \leq w \\
& \mathbf{t}\cdot\mathbf{x} + l + n \leq T \\
& \mathbf{x} \geq 0
\end{aligned}
$$




Variable sleeping time

# INEQUALITY CONSTRAINTS

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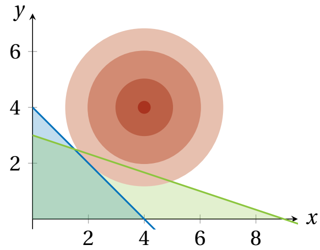


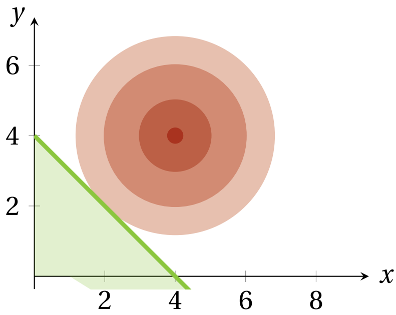

Variable sleeping time

Depending on $u, \mathbf{p}, w$, we may have $\mathbf{p} \cdot \mathbf{x} < w$ or $\mathbf{p} \cdot \mathbf{x} = w$

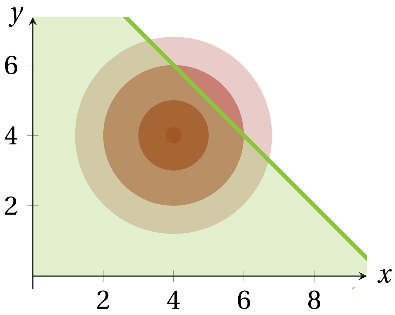
# INEQUALITY CONSTRAINTS

## INTUITION

Considering the general case the constraint can be either $g(\mathbf{x}^*) = c$ or $g(\mathbf{x}^*) < c$, or in other words the constraint might be **binding** or might be **slack**.




Binding constraint

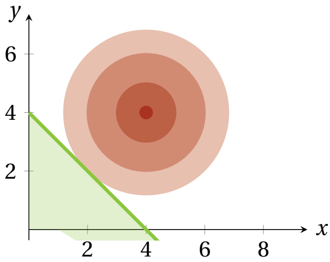



Slack constraint

# INEQUALITY CONSTRAINTS

## INTUITION

Consider the case when $g(\mathbf{x}^*) = c$, i.e. the constraint is **binding**.




Binding constraint

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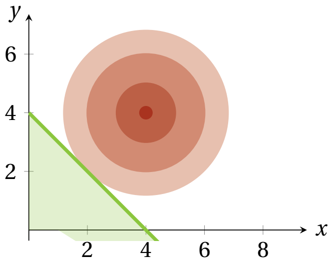


Binding constraint

If $g(\mathbf{x}^*) - c = 0$ and the constraint satisfies the FOC, then

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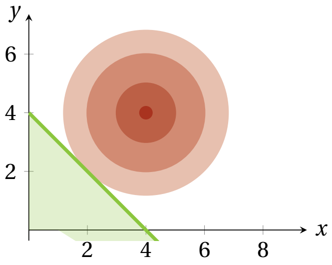


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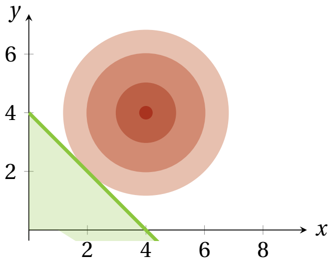


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If $g(\mathbf{x}^*) - c = 0$ and the constraint satisfies the FOC, then

- $\mathscr{L}_i(\mathbf{x}) = 0 \; \forall \, i$
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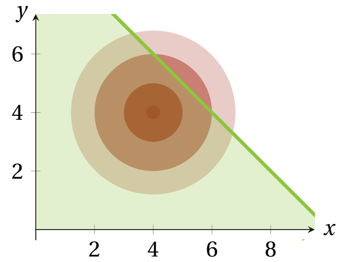
If $\lambda < 0$ then $\mathbf{x}^*$ would lay within the constraint, i.e. wouldn't have been a maximum in the first place. Contradiction.

# INEQUALITY CONSTRAINTS

## INTUITION

Consider the case when $g(\mathbf{x}^*) < c$, i.e. the constraint is **slack**.

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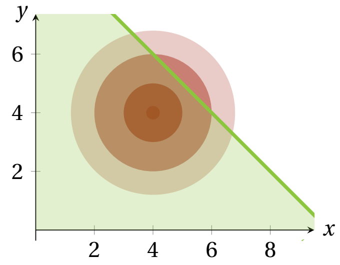
If $g(\mathbf{x}^*) < c$ , then

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If $g(\mathbf{x}^*) < c$ , then

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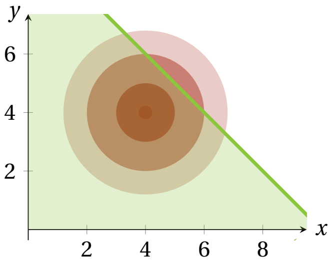


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# INEQUALITY CONSTRAINTS

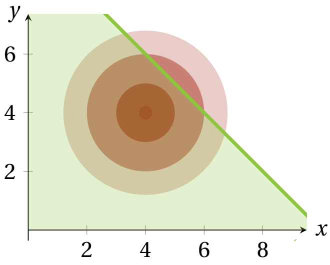
## INTUITION

Consider the case when $g(\mathbf{x}^*) < c$, i.e. the constraint is **slack**.

If $g(\mathbf{x}^*) < c$ , then

- $f_i(\mathbf{x}) = 0 \; \forall i$
- and $\lambda = 0$

In this case we can use $f_i(\mathbf{x})$ instead of $\mathcal{L}_i(\mathbf{x})$ because the solution under the constraint is the same as without the constraint, i.e. $f_i(\mathbf{x}) = \mathcal{L}_i(\mathbf{x})$.




Binding constraint

# INEQUALITY CONSTRAINTS

## INTUITION

Now we can combine the two cases and write the conditions as

- $\mathscr{L}_i(\mathbf{x}) = 0$ for $i = 1, \ldots, n$
- $\lambda_j \geq 0$
- $g_j(\mathbf{x}) - c_j \leq 0$
- $\lambda_j \left[ g_j(\mathbf{x}) - c_j \right] = 0$ for $j = 1, \ldots, n$

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The condition that either (i) $\lambda_j = 0$ and $g_j(\mathbf{x}^*) \leq c_j$ or (ii) $\lambda_j \geq 0$ and $g_j(\mathbf{x}^*) = c_j$ is called the **complementary slackness condition**.

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Note that we have not ruled out that both $\lambda_j = 0$ **and** $g_j(\mathbf{x}^*) = c$

# INEQUALITY CONSTRAINTS

## MINIMISATION PROBLEMS

What about the **minimisation** problems? we can convert a maximisation problem into one of minimisation flipping upside down the objective function $f(\mathbf{x})$ (multiplying it by $-1$), so that:

# INEQUALITY CONSTRAINTS

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- **REMARK 2:** for $g_j(\mathbf{x}) - c_j \geq 0$ then $\lambda_j \geq 0$

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- **REMARK 2:** for $g_j(\mathbf{x}) - c_j \geq 0$ then $\lambda_j \geq 0$
- **REMARK 3:** for Min and $g_j(\mathbf{x}) - c_j \geq 0$ then $\lambda_j \leq 0$

**DEFINITION:** let $f(\mathbf{x})$ and $g_j(\mathbf{x})$ be differentiable functions of $n$ variables and let $c_j$ for $j = 1, \ldots, m$ be numbers. Also define the function $\mathscr{L}$ of $n$ variables as:

$$\mathscr{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left( g_j(\mathbf{x}) - c_j \right) \text{ for all } \mathbf{x}$$

# INEQUALITY CONSTRAINTS

## KUHN-TUCKER CONDITIONS

The **Kuhn-Tucker conditions** of the problem:

$$\max_{\mathbf{x}} \quad f(\mathbf{x})$$
$$s.t. \quad g_j(\mathbf{x}) - c_j \leq 0 \text{ for } j = 1, \ldots, m$$

are:

- $\mathscr{L}_i(\mathbf{x}) = 0$ for $i = 1, \ldots, n$
- $\lambda_j \geq 0$
- $g_j(\mathbf{x}) \leq c_j$
- $\lambda_j \left[ g_j(\mathbf{x}) - c_j \right] = 0$ for $j = 1, \ldots, n$

The **SOLVING PROBLEM RECIPE**: consider the following problem:

$$\max_{\mathbf{x}} \quad f(\mathbf{x})$$
$$s.t. \quad g_j(\mathbf{x}) \leq c_j \text{ for } j = 1, \ldots, m$$

Where $\mathbf{x} = (x_1, \ldots, x_n)$

# INEQUALITY CONSTRAINTS

## KUHN-TUCKER CONDITIONS

**STEP 1**: Write down the Lagrangian

$$\mathscr{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left(g_j(\mathbf{x}) - c_j\right)$$

With $\lambda_1, \ldots, \lambda_m$ as the Lagrange multipliers with the $m$ constraints

# INEQUALITY CONSTRAINTS

## KUHN-TUCKER CONDITIONS

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With $\lambda_1, \ldots, \lambda_m$ as the Lagrange multipliers with the $m$ constraints

**STEP 2**: Equate all the first-order partial derivatives of $\mathscr{L}(\mathbf{x})$ to 0:

$$\frac{\partial \mathscr{L}(\mathbf{x})}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} = 0 \quad i = 1, \ldots, n$$

**STEP 3**: Impose the complementary slackness conditions:

$$\lambda_j \left[ g_j(\mathbf{x}) - c_j \right] = 0, \quad j = 1, \ldots, m$$

where either $\lambda_j > 0$ or $\lambda_j = 0$

# INEQUALITY CONSTRAINTS

## KUHN-TUCKER CONDITIONS

**STEP 3**: Impose the complementary slackness conditions:

$$\lambda_j \left[ g_j(\mathbf{x}) - c_j \right] = 0, \quad j = 1, \ldots, m$$

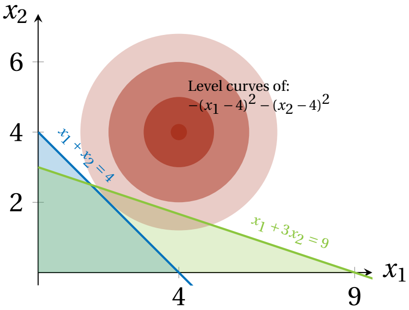
where either $\lambda_j > 0$ or $\lambda_j = 0$

**STEP 4**: Require $\mathbf{x}$ to satisfy the constraints:

$$g_j(\mathbf{x}) \leq c_j$$

**Example :**
Consider the problem

$$\max_{x_1,x_2} \quad -(x_1-4)^2-(x_2-4)^2$$
$$s.t. \quad x_1+x_2 \leq 4$$
$$x_1+3x_2 \leq 9$$



# INEQUALITY CONSTRAINTS

## KUHN-TUCKER CONDITIONS

**STEP 1**: Write down the Lagrangian

$$\mathscr{L}(x_1, x_2) = -(x_1 - 4)^2 - (x_2 - 4)^2 - \lambda_1 (x_1 + x_2 - 4) - \lambda_2 (x_1 + 3x_2 - 9)$$

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Hence arriving to a contradiction and being able to discard $(4,4)$

**CASE 2**: $\lambda_1 > 0$ and $\lambda_2 = 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 < 9$, The first constraint is binding but the sencond is not, the FOC become:

$$\left.\begin{aligned} \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_1} &= -2(x_1-4) - \lambda_1 = 0 \\ \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_2} &= -2(x_2-4) - \lambda_1 = 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 \quad (1)$$

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$$x_1 + x_1 = 4 \Rightarrow x_1^* = 2, x_2^* = 2$$

Checking the result against the other constraint $x_1 + 3x_2 \leq 9$:

$$2 + 3 \cdot 2 = 8 \leq 9$$

And then the point (2,2) is a candidate for a solution

**CASE 3**: $\lambda_1 = 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 < 4$ and $x_1 + 3x_2 = 9$, The first constraint is not binding but the sencond is, the FOC become:

$$\left.\begin{array}{lll} \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_1} & = -2(x_1 - 4) - \lambda_2 = 0 & \Rightarrow \lambda_2 = -2(x_1 - 4) \\ \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_2} & = -2(x_2 - 4) - 3\lambda_2 = 0 & \Rightarrow \lambda_2 = -\frac{2}{3}(x_2 - 4) \end{array}\right\}$$

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# INEQUALITY CONSTRAINTS

## KUHN-TUCKER CONDITIONS

Plugging (1) into (2)

$$\frac{1}{3}x_2 - \frac{8}{3} + 3x_2 = 9 \Rightarrow \frac{10}{3}x_2 = \frac{19}{3} \Rightarrow x_2^* = \frac{19}{10};\ x_1^* = \frac{33}{10}$$

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Checking the result against the other constraint $x_1 + x_2 \leq 4$:

$$\frac{33}{10} + \frac{19}{10} = \frac{52}{10} \leq 4 \ ↯$$

Hence arriving to a contradiction and being able to discard $\left(\frac{33}{10}, \frac{19}{10}\right)$

**CASE 4**: $\lambda_1 > 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 = 9$, now both constraints are binding, the FOC become:

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**CASE 4**: $\lambda_1 > 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 = 9$, now both constraints are binding, the FOC become:

$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial x_1} = -2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial x_2} = -2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$
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Solving the last two equations:

$$\left.\begin{array}{r} x_1 + x_2 = 4 \\ x_1 + 3x_2 = 9 \end{array}\right\} \Rightarrow (x_1^*, x_2^*) = \left(\frac{3}{2}, \frac{5}{2}\right)$$

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Then the first two equations become:

$$\left.\begin{aligned} 5 - \lambda_1 - \lambda_2 = 0 \\ 3 - \lambda_1 - 3\lambda_2 = 0 \end{aligned}\right\} \Rightarrow \lambda_1 = 6 \quad \text{and} \quad \lambda_2 = -1 \geq 0 \ \lightning$$

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Hence arriving to a contradiction and being able discard $\left(\frac{3}{2}, \frac{5}{2}\right)$

**SOLUTION:** so $(x_1^*, x_2^*, \lambda_1^*, \lambda_2^*) = (2, 2, 4, 0)$ is the single solution of the Kuhn-Tucker conditions. Hence the unique solution of the problem is $(x_1^*, x_2^*) = (2, 2)$